3/11/22

MATH 4030 Tutorial

Reminders:

- Assignment 4 due tonight 11:59pm.
- Q1 Assignment 3:

$$dF_p(v) = \langle \text{grad} F, v \rangle.$$

$$F(X(u,v)) \xrightarrow{\text{abusing notation}} F(u,v).$$

$$dF_p(v) = F_u(p)u'(0) + F_v(p)v'(0).$$

derivative of F wrt. 1st variable.

$$dF_p((av_1+bw_1)X_u + (av_2+bw_2)X_v)$$

$$= F_u(p)(av_1+bw_1) + F_v(p)(av_2+bw_2)$$

$v+w \in T_pM$. Take a curve γ s.t. $\gamma(0)=p$, $\gamma'(0) = \underbrace{v}_{\gamma_1'(0)} + \underbrace{w}_{\gamma_2'(0)}$.

$$dF_p(v+w) = \frac{d}{dt} F(\gamma(t))\Big|_{t=0}$$

$$= \ldots$$

A bit more on minimal surfaces

<u>Recall</u>: • M is minimal if $H \equiv 0$.

• Let $X(u,v)$ be a parameterization of M. X is isothermal if

$$|X_u| = |X_v| = \lambda, \quad \langle X_u, X_v \rangle = 0.$$

• M is minimal iff $X_{uu} + X_{vv} = 0$.

<u>Def</u>: If $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy Cauchy-Riemann equations if

$$\frac{\partial f}{\partial v} = \frac{\partial g}{\partial u}, \quad \frac{\partial f}{\partial u} = -\frac{\partial g}{\partial v}.$$

(From complex analysis, $h : \mathbb{C} \to \mathbb{C}$ is holomorphic if its real & imaginary part satisfy CR.).

Then they are harmonic, and they are called <u>harmonic conjugate</u>.

Let X, Y be isothermal parameterizations of minimal surfaces M, N, such that the component functions are pairwise harmonic conjugate, then M, N (X, Y) are called conjugate minimal surfaces.

1) Show that the helicoid and catenoid are conjugate minimal surfaces.

Catenoid: $X(u,v) = (\overset{f^1}{\cosh v \cos u}, \overset{f^2}{\cosh v \sin u}, \overset{f^3}{v})$

Helicoid: $Y(u,v) = (\underset{g^1}{\sinh v \sin u}, \underset{g^2}{-\sinh v \cos u}, \underset{g^3}{u})$.

Pf: Easy to check that these are isothermal.

$$\frac{\partial f^3}{\partial v} = 1 = \frac{\partial g^3}{\partial u}, \quad \frac{\partial f^3}{\partial u} = 0 = -0 = \frac{\partial g^3}{\partial v}. \quad ✓$$

$$\frac{\partial f^1}{\partial v} = \sinh v \cos u = \frac{\partial g^1}{\partial u}, \quad \frac{\partial f^1}{\partial u} = -\cosh v \sin u = -\frac{\partial g^1}{\partial v} \quad ✓$$

$$\frac{\partial f^2}{\partial v} = \sinh v \sin u = \frac{\partial g^2}{\partial u}, \quad \frac{\partial f^2}{\partial u} = \cosh v \cos u = -\frac{\partial g^2}{\partial v} \quad ✓$$

$$X_v = Y_u, \quad X_u = -Y_v.$$

2) Given two conjugate minimal surfaces X, Y, show that the surface

$$Z_t = (\cos t) X + (\sin t) Y$$

is minimal for all $t \in \mathbb{R}$.

Pf: $Z_u = \cos t X_u + \sin t Y_u$, $Z_v = \cos t X_v + \sin t Y_v$

$$\langle Z_u, Z_v \rangle = \cos^2 t \cancel{\langle X_u, X_v \rangle}^{0} + \sin t \cos t \langle X_u, Y_v \rangle + \sin t \cos t \langle Y_u, X_v \rangle + \sin^2 t \cancel{\langle Y_u, Y_v \rangle}^{0}$$

$Y_v = X_u$ $\qquad$ $Y_u = -X_v$

$$\stackrel{?}{=} \sin t \cos t \left(\cancel{|X_u|^2 - |X_v|^2}^{0} \right) = 0.$$

$$|Z_u|^2 = \langle Z_u, Z_u \rangle = \langle \cos t X_u + \sin t Y_u, \cos t X_u + \sin t Y_u \rangle$$

$$= \cos^2 t |X_u|^2 + 2 \sin t \cos t \cancel{\langle X_u, Y_u \rangle}^{0} + \sin^2 t |Y_u|^2$$

$$= \cos^2 t |X_u|^2 + \sin^2 t |\underset{-X_v}{Y_u}|^2$$

$$= \cos^2 t \lambda^2 + \sin^2 t \lambda^2 \qquad \text{where } \lambda^2 = |X_u|^2 = |X_v|^2$$

$= \lambda^2. = |Z_v|^2.$

similarly.

So Z is isothermal.

$Z_{uu} = \cos t\, X_{uu} + \sin t\, Y_{uu}$

$Z_{vv} = \cos t\, X_{vv} + \sin t\, Y_{vv}.$

$$Z_{uu} + Z_{vv} = \cos t(\cancel{X_{uu} + X_{vv}}^{0}) + \sin t(\cancel{Y_{uu} + Y_{vv}}^{0}).$$
$$= 0.$$

So Z_t is a minimal surface.

3) All Z_t above have the same coefficients of the 1^{st} fundamental form.

See above.

$$Z_t = \cos t \begin{pmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sinh v \sin u \\ -\sinh v \cos u \\ u \end{pmatrix}$$

is a 1-param. family of minimal surfaces that ($t=0$) ctsly deform the catenoid to the helicoid ($t=\frac{\pi}{2}$) and the intrinsic properties of Z_t

remain the same throughout. Another name: "associated family"